# §3. Reduction of M5-branes along $M_3$

Recall:

$$\begin{tikzcd}[column sep=small] & \text{6d } (2,0) \text{ th. of type } A_r \text{ on } M_3 \times \{S^1 \times D^2 \text{ or } L(k,1)_b\} \arrow[dl, "a)\ \text{along } M_3"'] \arrow[dr, "b)\ \text{along } S^1\times D^2"] & \\ \text{3d } \mathcal{N}=2 \text{ th } T[M_3] & & \text{complex CS on } M_3 \end{tikzcd}$$

In this paragraph we will follow reduction a) along $M_3$ via top.-twist to a 3d $\mathcal{N}=2$ theory and analyze the resulting MTC at low energies

## §3.1 Single fivebrane and Abelian MTC's

Consider M-theory on the spacetime

$$S^1 \times D^2 \times T^*M_3 \times \mathbb{R}^2$$

Consider M5-brane $\subset T^*M_3$ as a special Lagrangian : $\underset{\substack{\uparrow \\ \text{symplectic form}}}{\omega}\big|_{M_3} = 0$, $\mathrm{Im}(\underset{\substack{\uparrow \\ \text{hol. 3-form}}}{\Omega}\big|_{M_3}) = 0$

$\rightarrow$ fluctuations of M5 inside $T^*M_3$

indroduce local coordinates $x_i$ on $M_3$

$\rightarrow$ one-forms are locally $y_i \wedge dx_i$

coordinates along cotangent directions

$\rightarrow$ $M_3 = \{(x, 0) \in T^* M_3\}$

$\rightarrow$ deformation is described by

$$M_3' = \{(x, \underset{\substack{\uparrow \\ \text{one-form}}}{\lambda(x)})\} \subset T^* M_3$$

$M_3'$ must be also special Lagrangian to minimize energy!

In the local coordinates $(x, y) \in T^* M_3$, the symplectic form $\omega$ has the form:

$$\omega = dy_1 \wedge dx_1 + dy_2 \wedge dx_2 + dy_3 \wedge dx_3$$
$$= d(y_i \, dx_i)$$

$\rightarrow$ $\omega|_{M_3'} = d(\lambda)$ on deformed locus

$\Rightarrow$ $d(\lambda) \stackrel{!}{=} 0$ hence $\lambda$ is closed

Similarly, one can show for the imaginary part of the holomorphic 3-form: $\mathrm{Im}\left(\Omega\big|_{M_3'}\right) = d(*\lambda) \stackrel{!}{=} 0$

$\Rightarrow$ $\lambda$ must be harmonic form on $M_3$

$\longrightarrow$ $T_{M_3}\mathcal{M}\left(\begin{matrix}\text{special Lagr.}\\ \text{def. of } M_3\end{matrix}\right) \cong H^1(M_3;\mathbb{R})$

moduli space

$\longrightarrow$ deformations will appear as (real scalar) fields in effective theory $T[M_3]$

$\mathcal{N}=2$ SUSY $\longrightarrow$ real scalars have to be paired with other fields

Know M5-brane has 2-form $B$

$\longrightarrow$ reduction along 1-cycles of $M_3$ gives 3d gauge field $A_\mu$

$\longrightarrow$ together with scalars and fermions get $\mathcal{N}=2$ "vector multiplet":

$$V = (A_\mu, \lambda, \sigma, D)$$

→ get a gauge th. in 3d

**But!** Yang-Mills term irrelevant in IR!

→ can have Chern-Simons term

Suppose we have $U(1)$ CS-th. in 3d at level $k$, corresponding action:

$$S = \frac{k}{4\pi}\int d^3x\, d^4\theta\, V\Sigma\,, \quad \Sigma = \bar{D}^\alpha D_\alpha V$$

$$= \frac{k}{4\pi}\int \left(A\wedge dA - \bar{\lambda}\lambda + 2D\sigma\right)$$

→ have Wilson line observables along cycles $C$ in $S^1\times D^2$:

$$O_W = \exp\Big(i\underset{\text{charge}}{\underbrace{q}}\oint_C A\Big)$$

→ correlation functions are:

$$\left\langle \exp\left(iq_1\oint_{C_1} A\right)\exp\left(iq_2\oint_{C_2} A\right)\right\rangle \sim \exp\left(\frac{2\pi i q_1 q_2}{k}[C_1, C_2]\right)$$

where $[C_1, C_2]$ denotes the integer valued linking number between curves $C_i$

$\rightarrow$ correlation function vanishes

<u>iff</u> $q = 0 \mod k$

$\longrightarrow$ $q$ is valued in $\mathbb{Z}_K$ !

Back to our 3-manifold $M_3$:

gauge charges are captured by $H_1(M_3, \mathbb{Z})$

$\Longrightarrow$ CS-level $= K$ $\Longleftrightarrow$ $H_1(M_3, \mathbb{Z}) = \mathbb{Z}_K$

$\rightarrow$ torsion!

$$\int_{\alpha_q} B = \int_{\alpha_q} b \wedge A = \int_{q\alpha} b \wedge A = q \underbrace{\left( \int_\alpha b \right)}_{=1} A$$

where $q \in \mathbb{Z}_K$

<u>generalization:</u>

If gauge theory has $n$ $U(1)$ gauge fields with level matrix $K_{ij}$:

$$S = \int d^3x \sum_{i,j=1}^{n} \int d^4\theta \, \frac{K_{ij}}{4\pi} V_i \Sigma_j$$

$$= \frac{1}{4\pi} \int d^3x \; K^{ij} A_i \wedge dA_j + \cdots$$

then $H_1(M_3, \mathbb{Z})$ is generated by $n$ elements $T_i$ modulo relations defined by the image of $K_{ij}$:

$$H_1(M_3, \mathbb{Z}) \cong \bigoplus_{i=1}^{n} \mathbb{Z}[T_i] \Big/ (K_{ij} T_j = 0)$$

<u>Some facts about 3-manifolds:</u>

The 3-manifold $M_3$ can be realized as the boundary of a 4-manifold

$$M_3 = \partial M_4$$

To see this, note that in our setup

$$\partial(S^1 \times D^2) = T^2$$

$\rightarrow$ topologically $S^1 \times D^2 \sim T^2 \times \mathbb{R}_+$

Thus we can equivalently write

$$\underbrace{S^1 \times D^2 \times M_3}_{\text{wrapped by } M5} = T^2 \times M_4$$

where locally $M_4 \sim M_3 \times \mathbb{R}_+$

globally $M_4$ is a non-trivial 4-manifold such that generators $b_i$ of $T = H_2(M_4; \mathbb{Z})$ have intersection form $\langle b_i, b_j \rangle = K_{ij}$.

Example (Lens space):

The Lens space $L(p,q)$ is defined as

$$M_3 = L(p,q) = \left(S^3 = \{(z_1,z_2)\in\mathbb{C}^2 \mid |z_1|^2+|z_2|^2=1\}\right)/\mathbb{Z}_p$$

where

$$\mathbb{Z}_p : (z_1,z_2) \sim (e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2)$$

We assume $p$ and $q$ to be coprime integers

The matrix $K_{ij}$ is given by the following construction:

→ compute the continued fraction expansion for $-\frac{p}{q}$:

$$-\frac{p}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_n}}}$$

where $p > q > 0$ and restrict

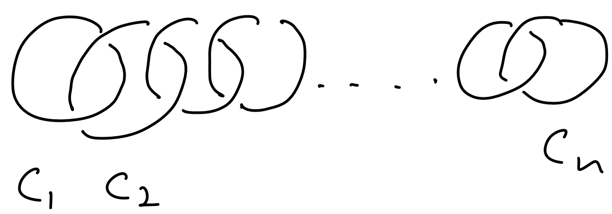
$a_i \leq -2$ for all $i=1,\ldots,n$

Then the 4-manifold $M_4$ with $\partial M_4 = L(p,q)$ has intersection form

$$K = \begin{pmatrix} a_1 & 1 & 0 & \cdots & & 0 \\ 1 & a_2 & 1 & \cdots & & 0 \\ 0 & 1 & \ddots & & & \vdots \\ \vdots & & \ddots & & & 0 \\ & & & \ddots & & 1 \\ 0 & \cdots & & 0 & 1 & a_n \end{pmatrix}$$

Interpretation:

$K$ can be thought of as the linking form on the following link chain




embedded in $S^3$

→ perform surgery by removing tubular neighborhood $N(C_i) \cong S^1 \times D^2$ of each link component and gluing it back after non-trivial diffeomorphism:

$$\phi : \underset{\partial N(C_i)}{\overset{\shortparallel}{T^2}} \longrightarrow T^2$$